

Question 8

Assume that

$$\lim_{x \rightarrow a} f(x) = L \quad \text{and} \quad f(x) \geq 0 \quad \text{for all } x \text{ near } a.$$

Prove that

$$L \geq 0.$$

Question 8 - Solution

We prove this by contradiction.

Assume, to the contrary, that

$$L < 0.$$

Then the positive number

$$\varepsilon = \frac{-L}{2}$$

satisfies $\varepsilon > 0$.

Since $\lim_{x \rightarrow a} f(x) = L$, there exists $\delta > 0$ such that whenever

$$0 < |x - a| < \delta,$$

we have

$$|f(x) - L| < \varepsilon.$$

This inequality implies

$$f(x) < L + \varepsilon = L + \frac{-L}{2} = \frac{L}{2}.$$

Because $L < 0$, we have

$$\frac{L}{2} < 0.$$

Thus, for all x with $0 < |x - a| < \delta$,

$$f(x) < 0.$$

This contradicts the assumption that $f(x) \geq 0$ for all x near a .

Therefore, our assumption that $L < 0$ must be false. We conclude that

$$L \geq 0.$$