

Question 7

Assume that

$$\lim_{x \rightarrow a} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = M,$$

and that

$$f(x) \leq g(x) \quad \text{for all } x \text{ near } a.$$

Prove that

$$L \leq M.$$

Question 7 - Solution

We argue by contradiction.

Suppose, to the contrary, that

$$L > M.$$

Then the positive number

$$\varepsilon = \frac{L - M}{2}$$

satisfies $\varepsilon > 0$.

Since $\lim_{x \rightarrow a} f(x) = L$, there exists $\delta_1 > 0$ such that whenever

$$0 < |x - a| < \delta_1,$$

we have

$$|f(x) - L| < \varepsilon.$$

This implies

$$f(x) > L - \varepsilon = \frac{L + M}{2}.$$

Similarly, since $\lim_{x \rightarrow a} g(x) = M$, there exists $\delta_2 > 0$ such that whenever

$$0 < |x - a| < \delta_2,$$

we have

$$|g(x) - M| < \varepsilon.$$

This implies

$$g(x) < M + \varepsilon = \frac{L + M}{2}.$$

Let

$$\delta = \min(\delta_1, \delta_2).$$

Then for all x satisfying $0 < |x - a| < \delta$, we have

$$f(x) > \frac{L + M}{2} \quad \text{and} \quad g(x) < \frac{L + M}{2}.$$

Thus,

$$f(x) > g(x),$$

which contradicts the assumption that $f(x) \leq g(x)$ near a .

Therefore, our assumption that $L > M$ must be false, and we conclude that

$$L \leq M.$$