

Question 4

Assume that $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$, with $M \neq 0$. Prove that

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

Question 4 - Solution

Let $\varepsilon > 0$. Since $g(x) \rightarrow M \neq 0$, choose $\delta_1 > 0$ such that $|g(x) - M| < |M|/2$ when $0 < |x - a| < \delta_1$. Then $|g(x)| > |M|/2$.

By the two assumed limits, choose $\delta_2, \delta_3 > 0$ to ensure respectively

$$|f(x) - L| < \frac{\varepsilon|M|}{4}, \quad |g(x) - M| < \frac{\varepsilon|M|^2}{4(1+|L|)}.$$

Take $\delta = \min(\delta_1, \delta_2, \delta_3)$. For $0 < |x - a| < \delta$,

$$\begin{aligned} \left| \frac{f(x)}{g(x)} - \frac{L}{M} \right| &\leq \frac{|f(x) - L|}{|g(x)|} + \frac{|L| |g(x) - M|}{|M| |g(x)|} \\ &\leq \frac{2|f(x) - L|}{|M|} + \frac{2|L| |g(x) - M|}{|M|^2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon|L|}{2(1+|L|)} < \varepsilon. \end{aligned}$$

(The first term makes the penultimate inequality strict even if $L = 0$.) This proves $\lim_{x \rightarrow a} f(x)/g(x) = L/M$.