

Question 3

Assume that $\lim_{x \rightarrow a} f(x) = L$. Prove that

$$\lim_{x \rightarrow a} |f(x)| = |L|.$$

Question 3 - Solution

Let $\varepsilon > 0$ be given. We must show that there exists $\delta > 0$ such that if

$$0 < |x - a| < \delta,$$

then

$$||f(x)| - |L|| < \varepsilon.$$

Recall the inequality

$$||u| - |v|| \leq |u - v|$$

for all real numbers u and v .

Apply this inequality with $u = f(x)$ and $v = L$:

$$||f(x)| - |L|| \leq |f(x) - L|.$$

Since $\lim_{x \rightarrow a} f(x) = L$, there exists $\delta > 0$ such that whenever

$$0 < |x - a| < \delta,$$

we have

$$|f(x) - L| < \varepsilon.$$

Combining the inequalities gives

$$||f(x)| - |L|| \leq |f(x) - L| < \varepsilon.$$

Thus, whenever $0 < |x - a| < \delta$,

$$||f(x)| - |L|| < \varepsilon.$$

Therefore,

$$\lim_{x \rightarrow a} |f(x)| = |L|.$$