

Question 2

Assume that $\lim_{x \rightarrow a} f(x) = L$ and that c is a constant. Prove that

$$\lim_{x \rightarrow a} (cf(x)) = cL.$$

Question 2 - Solution

Let $\varepsilon > 0$ be given. We must show that there exists $\delta > 0$ such that if

$$0 < |x - a| < \delta,$$

then

$$|cf(x) - cL| < \varepsilon.$$

Start by factoring out the constant:

$$|cf(x) - cL| = |c| \cdot |f(x) - L|.$$

Since $\lim_{x \rightarrow a} f(x) = L$, there exists $\delta > 0$ such that whenever

$$0 < |x - a| < \delta,$$

we have

$$|f(x) - L| < \frac{\varepsilon}{|c|},$$

provided $c \neq 0$.

Substitute this into the inequality:

$$|cf(x) - cL| = |c| \cdot |f(x) - L| < |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon.$$

Thus, whenever $0 < |x - a| < \delta$, we have

$$|cf(x) - cL| < \varepsilon.$$

Therefore,

$$\lim_{x \rightarrow a} (cf(x)) = cL.$$

If $c = 0$, then $cf(x) = 0$ for all x , and the result holds trivially.