

Question 1

Assume $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$. Prove that

$$\lim_{x \rightarrow a} (f(x)g(x)) = LM.$$

Question 1 - Solution

Let $\varepsilon > 0$ be given.

We must show there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then

$$|f(x)g(x) - LM| < \varepsilon.$$

Start by adding and subtracting $Lg(x)$:

$$f(x)g(x) - LM = f(x)g(x) - Lg(x) + Lg(x) - LM = g(x)(f(x) - L) + L(g(x) - M).$$

Take absolute values and use the triangle inequality:

$$|f(x)g(x) - LM| \leq |g(x)| |f(x) - L| + |L| |g(x) - M|.$$

To control $|g(x)|$, use the fact that $g(x) \rightarrow M$.

Since $\lim_{x \rightarrow a} g(x) = M$, there exists $\delta_1 > 0$ such that if $0 < |x - a| < \delta_1$, then

$$|g(x) - M| < 1.$$

Then, for $0 < |x - a| < \delta_1$,

$$|g(x)| = |g(x) - M + M| \leq |g(x) - M| + |M| < 1 + |M|.$$

Now choose $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$, then

$$|f(x) - L| < \frac{\varepsilon}{2(1 + |M|)}.$$

This is possible because $\lim_{x \rightarrow a} f(x) = L$.

Next choose $\delta_3 > 0$ such that if $0 < |x - a| < \delta_3$, then

$$|g(x) - M| < \frac{\varepsilon}{2(1 + |L|)}.$$

This is possible because $\lim_{x \rightarrow a} g(x) = M$.

Let

$$\delta = \min(\delta_1, \delta_2, \delta_3).$$

Then whenever $0 < |x - a| < \delta$, we have simultaneously:

$$|g(x)| < 1 + |M|, \quad |f(x) - L| < \frac{\varepsilon}{2(1 + |M|)}, \quad |g(x) - M| < \frac{\varepsilon}{2(1 + |L|)}.$$

Substitute into the estimate:

$$|f(x)g(x) - LM| \leq |g(x)| |f(x) - L| + |L| |g(x) - M|.$$

For the first term:

$$|g(x)| |f(x) - L| < (1 + |M|) \cdot \frac{\varepsilon}{2(1 + |M|)} = \frac{\varepsilon}{2}.$$

For the second term:

$$|L| |g(x) - M| \leq \frac{\varepsilon |L|}{2(1 + |L|)} < \frac{\varepsilon}{2}.$$

Therefore,

$$|f(x)g(x) - LM| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that

$$\lim_{x \rightarrow a} (f(x)g(x)) = LM.$$