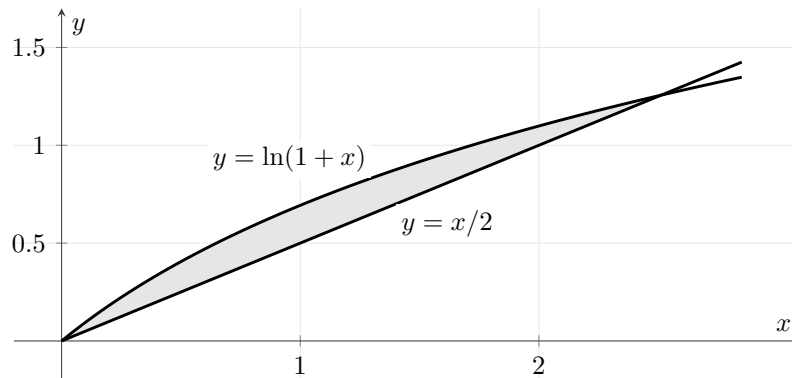


Question 7

The base of a solid is the region bounded by

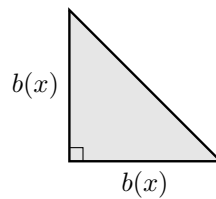
$$y = \ln(x + 1) \quad \text{and} \quad y = \frac{x}{2},$$

between their points of intersection. Cross-sections perpendicular to the x -axis are right triangles whose height is equal to the base. Find the volume of the solid.



Shaded base region in the xy -plane.

Cross-section perpendicular to the x -axis



Schematic cross-section; dimensions vary with x .

Question 7 – Solution

1. Define the intersection bounds.

Let $b(x) = \ln(1+x) - x/2$. Since $b'(x) = (1-x)/(2(1+x))$, b increases to $x = 1$ and then decreases. Its zeros are 0 and a unique $a > 1$ with

$$\ln(1+a) = a/2, \quad a \approx 2.512862417.$$

This equation defines a exactly; $b(x) > 0$ on $(0, a)$.

2. Find the triangular area and set up the volume.

Both perpendicular legs have length $b(x)$, so

$$V = \frac{1}{2} \int_0^a \left(\ln(1+x) - \frac{x}{2} \right)^2 dx.$$

3. Expand and find the needed antiderivatives.

Write $L = \ln(1+x)$ and $u = 1+x$. The square is $L^2 - xL + x^2/4$. Integration by parts gives

$$\int L^2 dx = u(L^2 - 2L + 2), \quad \int xL dx = \frac{x^2}{2}L - \frac{x^2}{4} + \frac{x}{2} - \frac{L}{2}.$$

For the first identity, use $\int \ln^2 u du = u \ln^2 u - 2 \int \ln u du$.

4. Combine the terms and apply the endpoints.

An antiderivative of the squared expression is

$$H(x) = uL^2 - \left(u + \frac{u^2}{2}\right)L + u + \frac{u^2}{4} + \frac{x^3}{12}.$$

At $x = 0$, $H(0) = 5/4$. Substituting $u = 1+a$ and $L = a/2$ gives

$$H(a) - H(0) = \frac{a^3}{12} - \frac{a^2}{2} + \frac{3a}{4} = \frac{a(a-3)^2}{12}.$$

5. Include the triangular-area factor.

$$V = \frac{1}{2}[H(a) - H(0)] = \boxed{\frac{a(a-3)^2}{24}} \approx 0.024846244.$$

All volumes are in cubic units.