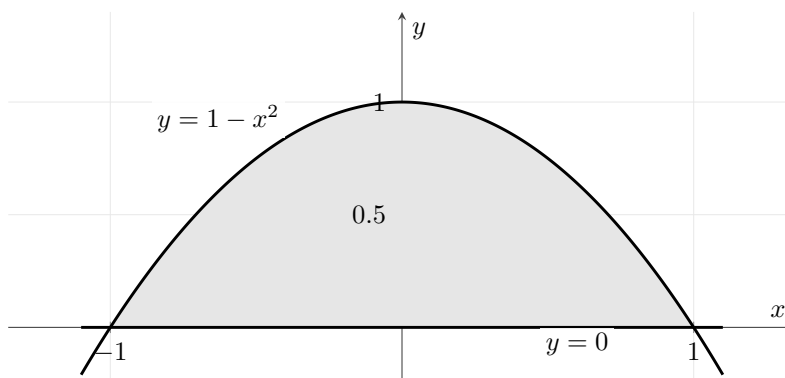


Question 4

The base of a solid is the region enclosed by

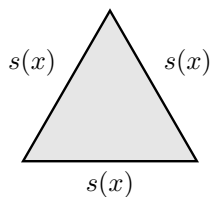
$$y = 1 - x^2 \quad \text{and} \quad y = 0$$

between their intersection points. Cross-sections perpendicular to the x -axis are equilateral triangles. Find the volume of the solid.



Shaded base region in the xy -plane.

Cross-section perpendicular to the x -axis



Schematic cross-section; dimensions vary with x .

Question 4 – Solution**1. Find the bounds and side length.**

The intersections satisfy $1 - x^2 = 0$, so $x = -1, 1$. A vertical base segment gives

$$s(x) = 1 - x^2 \quad (-1 \leq x \leq 1).$$

2. Find the area of an equilateral triangle.

The altitude is $\sqrt{s^2 - (s/2)^2} = \sqrt{3} s/2$. Hence

$$A = \frac{1}{2}s \left(\frac{\sqrt{3}}{2}s \right) = \frac{\sqrt{3}}{4}s^2, \quad A(x) = \frac{\sqrt{3}}{4}(1 - x^2)^2.$$

3. Set up the volume and expand.

$$V = \frac{\sqrt{3}}{4} \int_{-1}^1 (1 - x^2)^2 dx = \frac{\sqrt{3}}{4} \int_{-1}^1 (1 - 2x^2 + x^4) dx.$$

4. Use symmetry and integrate.

The integrand is even, so

$$V = \frac{\sqrt{3}}{2} \int_0^1 (1 - 2x^2 + x^4) dx = \frac{\sqrt{3}}{2} \left[x - \frac{2x^3}{3} + \frac{x^5}{5} \right]_0^1.$$

5. Evaluate and simplify.

$$V = \frac{\sqrt{3}}{2} \left(1 - \frac{2}{3} + \frac{1}{5} \right) = \frac{\sqrt{3}}{2} \left(\frac{15 - 10 + 3}{15} \right) = \boxed{\frac{4\sqrt{3}}{15}}.$$

All volumes are in cubic units.