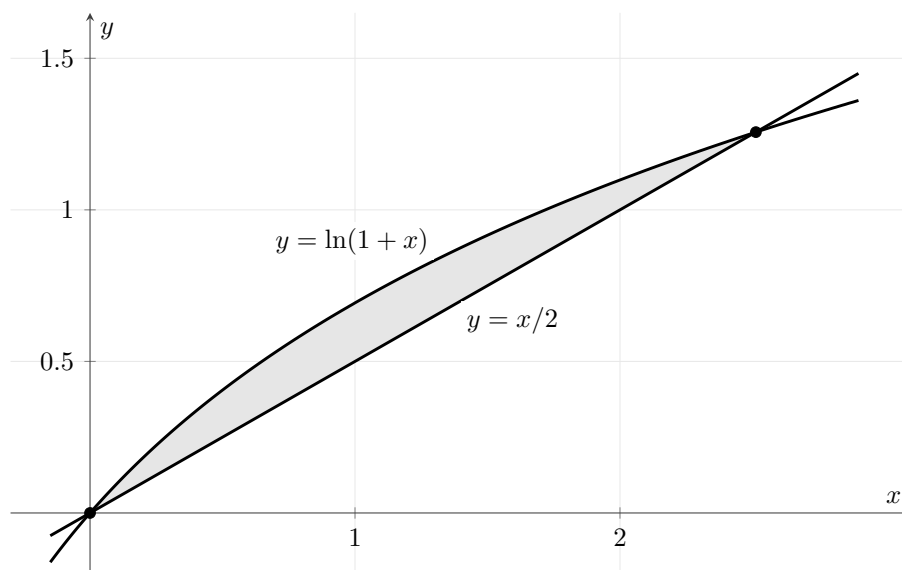


Question 9

Find the exact area of the region enclosed by the curves

$$y = \ln(1 + x) \quad \text{and} \quad y = \frac{x}{2},$$

between their points of intersection.



The shaded region is the area to be found.

Question 9 – Solution

1. Describe the intersections.

Let $h(x) = \ln(1+x) - x/2$, with domain $x > -1$. Then

$$h'(x) = \frac{1-x}{2(1+x)}.$$

So h increases up to $x = 1$ and decreases thereafter. Since $h(0) = 0$, $h(1) > 0$, and $h(x) \rightarrow -\infty$ as $x \rightarrow \infty$, the only roots are 0 and a unique number $a > 1$ satisfying

$$\ln(1+a) = \frac{a}{2}, \quad a \approx 2.512862417.$$

The nonzero root has no elementary closed form; this equation defines a exactly.

2. Set up the area.

On $(0, a)$, $h(x) > 0$, so the logarithm is above the line.

$$A = \int_0^a \left(\ln(1+x) - \frac{x}{2} \right) dx.$$

3. Find an antiderivative.

With $u = 1+x$, integration by parts gives

$$\int \ln(1+x) dx = (1+x) \ln(1+x) - x + C.$$

Also, $\int (x/2) dx = x^2/4$. At $x = 0$, all terms below are zero.

4. Evaluate, then substitute $\ln(1+a) = a/2$.

$$\begin{aligned} A &= \left[(1+x) \ln(1+x) - x - \frac{x^2}{4} \right]_0^a \\ &= (1+a) \frac{a}{2} - a - \frac{a^2}{4} \\ &= \boxed{\frac{a(a-2)}{4}} \approx 0.322188173. \end{aligned}$$

5. Interpret the exact answer.

The boxed expression is exact with a defined above; using its numerical approximation gives 0.322188173 square units.