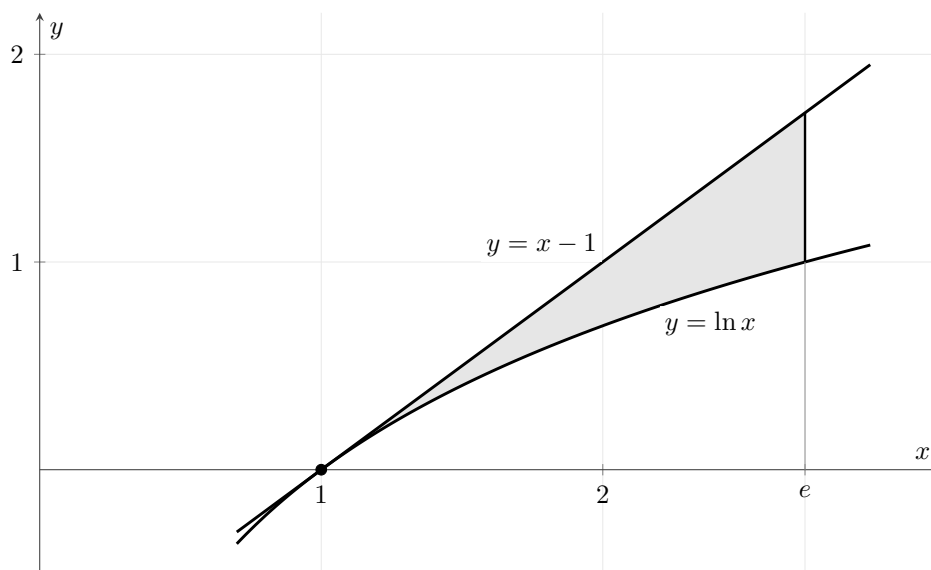


Question 4

Find the exact area of the region between the curves

$$y = \ln x \quad \text{and} \quad y = x - 1,$$

between $x = 1$ and $x = e$.



The shaded region is the area to be found.

Question 4 – Solution

1. Identify the upper curve on $[1, e]$.

Let $h(x) = x - 1 - \ln x$. Then $h(1) = 0$ and

$$h'(x) = 1 - \frac{1}{x} \geq 0 \quad (1 \leq x \leq e).$$

Thus $x - 1 \geq \ln x$ throughout the interval.

2. Set up the area.

$$A = \int_1^e (x - 1 - \ln x) dx.$$

3. Find the logarithm antiderivative.

Integration by parts, with $u = \ln x$ and $dv = dx$, gives

$$\int \ln x dx = x \ln x - \int 1 dx = x \ln x - x.$$

Here $du = dx/x$ and $v = x$. Combine the terms: $x^2/2 - x - (x \ln x - x) = x^2/2 - x \ln x$.

4. Apply the upper and lower bounds.

$$\begin{aligned} A &= \left[\frac{x^2}{2} - x - (x \ln x - x) \right]_1^e \\ &= \left[\frac{x^2}{2} - x \ln x \right]_1^e \\ &= \left(\frac{e^2}{2} - e \right) - \frac{1}{2} = \boxed{\frac{e^2}{2} - e - \frac{1}{2}}. \end{aligned}$$

5. Check the result.

Since $\ln e = 1$ and $\ln 1 = 0$, the endpoint terms above give an area of approximately 0.4762 square units, which is positive.