

Question 10

Find the average value of the function

$$f(x) = x e^{-x}$$

on the interval $[0, 2]$.

Question 10 - Solution

The average value of a function on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Here,

$$f_{\text{avg}} = \frac{1}{2} \int_0^2 x e^{-x} dx.$$

Evaluate the integral using integration by parts.

Let

$$u = x, \quad dv = e^{-x} dx.$$

Then

$$du = dx, \quad v = -e^{-x}.$$

Apply integration by parts:

$$\int x e^{-x} dx = -x e^{-x} + \int e^{-x} dx = -x e^{-x} - e^{-x}.$$

Thus,

$$\int x e^{-x} dx = -(x+1)e^{-x}.$$

Evaluate from 0 to 2:

$$\left[-(x+1)e^{-x} \right]_0^2 = -3e^{-2} - (-1) = 1 - 3e^{-2}.$$

Now compute the average value:

$$f_{\text{avg}} = \frac{1}{2} (1 - 3e^{-2}).$$

$$\boxed{\frac{1}{2} (1 - 3e^{-2})}$$