

Question 9

Find the average value of the function

$$f(x) = \sqrt{x^2 + 1}$$

on the interval $[-1, 1]$.

Question 9 - Solution

The average value of a function on an interval $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Here,

$$f_{\text{avg}} = \frac{1}{2} \int_{-1}^1 \sqrt{x^2 + 1} dx.$$

Notice that $\sqrt{x^2 + 1}$ is an even function, so

$$\int_{-1}^1 \sqrt{x^2 + 1} dx = 2 \int_0^1 \sqrt{x^2 + 1} dx.$$

Thus,

$$f_{\text{avg}} = \int_0^1 \sqrt{x^2 + 1} dx.$$

Evaluate the integral using trigonometric substitution. Let

$$x = \tan \theta, \quad dx = \sec^2 \theta d\theta.$$

When $x = 0$, $\theta = 0$. When $x = 1$, $\theta = \frac{\pi}{4}$.

Substitute:

$$\int_0^1 \sqrt{x^2 + 1} dx = \int_0^{\pi/4} \sec \theta \cdot \sec^2 \theta d\theta = \int_0^{\pi/4} \sec^3 \theta d\theta.$$

Use the standard result

$$\int \sec^3 \theta d\theta = \frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|).$$

Apply the limits:

$$\frac{1}{2} (\sec \theta \tan \theta + \ln(\sec \theta + \tan \theta)) \Big|_0^{\pi/4}.$$

Evaluate:

$$\begin{aligned} \sec\left(\frac{\pi}{4}\right) &= \sqrt{2}, & \tan\left(\frac{\pi}{4}\right) &= 1, \\ \sec(0) &= 1, & \tan(0) &= 0. \end{aligned}$$

Thus,

$$f_{\text{avg}} = \frac{1}{2} \left(\sqrt{2} + \ln(1 + \sqrt{2}) \right).$$

$$\boxed{\frac{1}{2} \left(\sqrt{2} + \ln(1 + \sqrt{2}) \right)}$$