

Question 7

Find the average value of the function

$$f(x) = \sqrt{1 - x^2}$$

on the interval $[-1, 1]$.

Question 7 - Solution

The average value of a function on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Here,

$$f_{\text{avg}} = \frac{1}{2} \int_{-1}^1 \sqrt{1-x^2} dx.$$

Method 1: Geometry

The graph of

$$y = \sqrt{1-x^2}$$

is the upper half of a circle of radius 1 centered at the origin.

The area under this curve from $x = -1$ to $x = 1$ is the area of a semicircle:

$$\frac{1}{2} \pi (1)^2 = \frac{\pi}{2}.$$

Therefore,

$$f_{\text{avg}} = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}.$$

Method 2: Trigonometric Substitution

Evaluate the integral directly.

Let

$$x = \sin \theta, \quad dx = \cos \theta d\theta.$$

Change the limits. When $x = -1$, $\theta = -\frac{\pi}{2}$. When $x = 1$, $\theta = \frac{\pi}{2}$.

Substitute:

$$\int_{-1}^1 \sqrt{1-x^2} dx = \int_{-\pi/2}^{\pi/2} \cos^2 \theta d\theta.$$

Use the identity

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}.$$

Then

$$\int_{-\pi/2}^{\pi/2} \cos^2 \theta d\theta = \frac{1}{2} \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta) d\theta.$$

Integrate:

$$\frac{1}{2} \left[\theta + \frac{1}{2} \sin(2\theta) \right]_{-\pi/2}^{\pi/2}.$$

Since $\sin(\pi) = \sin(-\pi) = 0$, this becomes

$$\frac{1}{2} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right) = \frac{\pi}{2}.$$

Now compute the average value:

$$f_{\text{avg}} = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}.$$

$$\boxed{\frac{\pi}{4}}$$