

Question 5

Find the average value of the function

$$f(x) = \frac{x}{1+x^2}$$

on the interval $[0, 1]$.

Question 5 - Solution

The average value of a function f on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Here $a = 0$ and $b = 1$, so

$$f_{\text{avg}} = \int_0^1 \frac{x}{1+x^2} dx.$$

Evaluate the integral using substitution.

Let

$$u = 1 + x^2.$$

Then

$$du = 2x dx \quad \Rightarrow \quad x dx = \frac{1}{2} du.$$

Change the limits. When $x = 0$, $u = 1$. When $x = 1$, $u = 2$.

Substitute:

$$\int_0^1 \frac{x}{1+x^2} dx = \frac{1}{2} \int_1^2 \frac{1}{u} du.$$

Integrate:

$$\frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln u.$$

Apply the limits:

$$\frac{1}{2} \ln u \Big|_1^2 = \frac{1}{2} \ln 2.$$

$$\boxed{\frac{1}{2} \ln 2}$$