

Question 3

Find the average value of the function

$$f(x) = x^2 \cos x$$

on the interval $[0, \pi]$.

Question 3 - Solution

The average value of a function f on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Here,

$$f_{\text{avg}} = \frac{1}{\pi} \int_0^\pi x^2 \cos x dx.$$

Evaluate the integral using integration by parts.

Let

$$u = x^2, \quad dv = \cos x dx.$$

Then

$$du = 2x dx, \quad v = \sin x.$$

Apply integration by parts:

$$\int x^2 \cos x dx = x^2 \sin x - \int 2x \sin x dx.$$

Apply integration by parts again to the remaining integral. Let

$$u = 2x, \quad dv = \sin x dx.$$

Then

$$du = 2 dx, \quad v = -\cos x.$$

Thus,

$$\int 2x \sin x dx = -2x \cos x + 2 \int \cos x dx = -2x \cos x + 2 \sin x.$$

Substitute back:

$$\int x^2 \cos x dx = x^2 \sin x + 2x \cos x - 2 \sin x.$$

Evaluate from 0 to π :

$$\left[x^2 \sin x + 2x \cos x - 2 \sin x \right]_0^\pi.$$

At $x = \pi$:

$$\pi^2 \sin \pi + 2\pi \cos \pi - 2 \sin \pi = -2\pi.$$

At $x = 0$:

$$0 + 0 - 0 = 0.$$

Thus,

$$\int_0^\pi x^2 \cos x dx = -2\pi.$$

Divide by the interval length:

$$f_{\text{avg}} = \frac{1}{\pi}(-2\pi) = -2.$$

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