

**Question 2**

Find the average value of the function

$$f(x) = \ln(1 + x^2)$$

on the interval  $[-1, 1]$ .

**Question 2 - Solution**

The average value of a function  $f$  on  $[a, b]$  is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Here  $a = -1$  and  $b = 1$ , so

$$f_{\text{avg}} = \frac{1}{2} \int_{-1}^1 \ln(1+x^2) dx.$$

Notice that  $\ln(1+x^2)$  is an even function, since

$$\ln(1+(-x)^2) = \ln(1+x^2).$$

Therefore,

$$\int_{-1}^1 \ln(1+x^2) dx = 2 \int_0^1 \ln(1+x^2) dx.$$

Thus,

$$f_{\text{avg}} = \int_0^1 \ln(1+x^2) dx.$$

Evaluate the integral using integration by parts. Let

$$u = \ln(1+x^2), \quad dv = dx.$$

Then

$$du = \frac{2x}{1+x^2} dx, \quad v = x.$$

Apply integration by parts:

$$\int \ln(1+x^2) dx = x \ln(1+x^2) - \int \frac{2x^2}{1+x^2} dx.$$

Rewrite the integrand:

$$\frac{2x^2}{1+x^2} = 2 - \frac{2}{1+x^2}.$$

Thus,

$$\int \ln(1+x^2) dx = x \ln(1+x^2) - \int \left(2 - \frac{2}{1+x^2}\right) dx.$$

Integrate:

$$x \ln(1+x^2) - 2x + 2 \arctan x.$$

Evaluate from 0 to 1:

$$\left[ x \ln(1+x^2) - 2x + 2 \arctan x \right]_0^1 = \ln 2 - 2 + \frac{\pi}{2}.$$

Therefore,

$$\boxed{\ln 2 - 2 + \frac{\pi}{2}}$$