

Question 1

Find the average value of the function

$$f(x) = x\sqrt{1+x^2}$$

on the interval $[0, 2]$.

Question 1 - Solution

The average value of a function f on an interval $[a, b]$ is defined by

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Here, $a = 0$ and $b = 2$, so

$$f_{\text{avg}} = \frac{1}{2} \int_0^2 x \sqrt{1+x^2} dx.$$

Evaluate the integral using substitution. Let

$$u = 1 + x^2.$$

Then

$$du = 2x dx \quad \Rightarrow \quad x dx = \frac{1}{2} du.$$

Change the limits. When $x = 0$, $u = 1$. When $x = 2$, $u = 5$.

Substitute:

$$\int_0^2 x \sqrt{1+x^2} dx = \frac{1}{2} \int_1^5 u^{1/2} du.$$

Integrate:

$$\frac{1}{2} \int u^{1/2} du = \frac{1}{2} \cdot \frac{2}{3} u^{3/2} = \frac{1}{3} u^{3/2}.$$

Apply the limits:

$$\left. \frac{1}{3} u^{3/2} \right|_1^5 = \frac{1}{3} (5^{3/2} - 1).$$

Now divide by the interval length:

$$f_{\text{avg}} = \frac{1}{2} \cdot \frac{1}{3} (5^{3/2} - 1) = \frac{1}{6} (5^{3/2} - 1).$$

$$\boxed{\frac{1}{6} (5^{3/2} - 1)}$$