

Question 10

Evaluate the definite integral

$$\int_0^1 \sqrt{1-x^2} \arcsin x \, dx.$$

Question 10 - Solution

This integral is interesting because it mixes an inverse trigonometric function with an algebraic expression, suggesting a trigonometric substitution.

Let

$$x = \sin \theta.$$

Then

$$dx = \cos \theta \, d\theta, \quad \sqrt{1-x^2} = \cos \theta.$$

Change the limits of integration. When $x = 0$, $\theta = 0$. When $x = 1$, $\theta = \frac{\pi}{2}$.

Substitute into the integral:

$$\int_0^1 \sqrt{1-x^2} \arcsin x \, dx = \int_0^{\pi/2} \theta \cos^2 \theta \, d\theta.$$

Use the identity

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}.$$

Then

$$\int_0^{\pi/2} \theta \cos^2 \theta \, d\theta = \frac{1}{2} \int_0^{\pi/2} \theta \, d\theta + \frac{1}{2} \int_0^{\pi/2} \theta \cos(2\theta) \, d\theta.$$

The first integral is

$$\frac{1}{2} \int_0^{\pi/2} \theta \, d\theta = \frac{1}{2} \cdot \frac{(\pi/2)^2}{2} = \frac{\pi^2}{16}.$$

For the second integral, use integration by parts. Let

$$u = \theta, \quad dv = \cos(2\theta) \, d\theta.$$

Then

$$du = d\theta, \quad v = \frac{1}{2} \sin(2\theta).$$

Thus,

$$\frac{1}{2} \int_0^{\pi/2} \theta \cos(2\theta) \, d\theta = \frac{1}{2} \left[\frac{1}{2} \theta \sin(2\theta) - \frac{1}{2} \int \sin(2\theta) \, d\theta \right]_0^{\pi/2}.$$

Since $\sin(\pi) = 0$ and $\sin(0) = 0$, the boundary term vanishes. Also,

$$\int \sin(2\theta) \, d\theta = -\frac{1}{2} \cos(2\theta).$$

Therefore,

$$\frac{1}{2} \int_0^{\pi/2} \theta \cos(2\theta) \, d\theta = \frac{1}{8} [\cos(2\theta)]_0^{\pi/2} = \frac{1}{8} (-1 - 1) = -\frac{1}{4}.$$

Combine results:

$$\frac{\pi^2}{16} - \frac{1}{4}.$$

$$\boxed{\frac{\pi^2}{16} - \frac{1}{4}}$$