

Question 9

Evaluate the definite integral

$$\int_0^1 \frac{\sqrt{1-x}}{1+x} dx.$$

Question 9 - Solution

This integral becomes simpler if we eliminate the square root by substitution.

Let

$$u = \sqrt{1-x}.$$

Then

$$x = 1 - u^2, \quad dx = -2u \, du.$$

Change the limits of integration. When $x = 0$, $u = 1$. When $x = 1$, $u = 0$.

Substitute into the integral:

$$\int_0^1 \frac{\sqrt{1-x}}{1+x} dx = \int_1^0 \frac{u}{1+(1-u^2)} (-2u) du.$$

Simplify:

$$\int_1^0 \frac{-2u^2}{2-u^2} du = \int_0^1 \frac{2u^2}{2-u^2} du.$$

Rewrite the integrand:

$$\frac{2u^2}{2-u^2} = -2 + \frac{4}{2-u^2}.$$

Thus,

$$\int_0^1 \left(-2 + \frac{4}{2-u^2} \right) du.$$

Integrate term by term:

$$\begin{aligned} \int -2 \, du &= -2u, \\ \int \frac{4}{2-u^2} \, du &= 2 \int \frac{1}{1-(u/\sqrt{2})^2} \, du. \end{aligned}$$

Use the standard result

$$\int \frac{1}{1-a^2} da = \operatorname{artanh}(a).$$

Thus,

$$2 \int \frac{1}{1-(u/\sqrt{2})^2} du = 2\sqrt{2} \operatorname{artanh}\left(\frac{u}{\sqrt{2}}\right).$$

Evaluate from 0 to 1:

$$\left. -2u + 2\sqrt{2} \operatorname{artanh}\left(\frac{u}{\sqrt{2}}\right) \right|_0^1.$$

Final answer:

$$-2 + 2\sqrt{2} \operatorname{artanh}\left(\frac{1}{\sqrt{2}}\right)$$