

Question 3

Evaluate the definite integral

$$\int_{-1}^1 x^2 \sqrt{1-x^2} \, dx.$$

Question 3 - Solution

First notice a symmetry property. The function

$$x^2\sqrt{1-x^2}$$

is an *even function* because replacing x by $-x$ does not change its value. Therefore,

$$\int_{-1}^1 x^2\sqrt{1-x^2} dx = 2 \int_0^1 x^2\sqrt{1-x^2} dx.$$

Now focus on the integral from 0 to 1.

Use a trigonometric substitution. Let

$$x = \sin \theta, \quad dx = \cos \theta d\theta.$$

When $x = 0$, $\theta = 0$, and when $x = 1$, $\theta = \frac{\pi}{2}$.

Substitute into the integral:

$$\int_0^1 x^2\sqrt{1-x^2} dx = \int_0^{\pi/2} (\sin^2 \theta)(\cos \theta)(\cos \theta) d\theta = \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta d\theta.$$

Use the identity

$$\sin^2 \theta \cos^2 \theta = \frac{1}{4} \sin^2(2\theta).$$

Then

$$\int_0^{\pi/2} \sin^2 \theta \cos^2 \theta d\theta = \frac{1}{4} \int_0^{\pi/2} \sin^2(2\theta) d\theta.$$

Apply the identity

$$\sin^2 u = \frac{1 - \cos(2u)}{2}.$$

Thus,

$$\frac{1}{4} \int_0^{\pi/2} \frac{1 - \cos(4\theta)}{2} d\theta = \frac{1}{8} \left[\theta - \frac{1}{4} \sin(4\theta) \right]_0^{\pi/2}.$$

Evaluate:

$$\frac{1}{8} \left(\frac{\pi}{2} - 0 \right) = \frac{\pi}{16}.$$

Finally, multiply by 2 to account for the symmetry:

$$\int_{-1}^1 x^2\sqrt{1-x^2} dx = 2 \cdot \frac{\pi}{16} = \boxed{\frac{\pi}{8}}.$$