

Question 9

Let

$$f(x) = \begin{cases} 2x, & 0 \leq x \leq 1, \\ 2, & 1 < x \leq 2. \end{cases}$$

Using the definition of the definite integral, express

$$\int_0^2 f(x) \, dx$$

as the limit of a Riemann sum, and evaluate the limit.

Question 9 - Solution

Partition the interval $[0, 2]$ into n equal subintervals. Each subinterval has width

$$\Delta x = \frac{2}{n}.$$

Using right endpoints, the i th sample point is

$$x_i = \frac{2i}{n}.$$

The Riemann sum is

$$\sum_{i=1}^n f(x_i) \Delta x = \frac{2}{n} \sum_{i=1}^n f\left(\frac{2i}{n}\right).$$

Split the sum according to where the function changes behavior. The condition $x_i \leq 1$ corresponds to

$$\frac{2i}{n} \leq 1 \quad \Rightarrow \quad i \leq \frac{n}{2}.$$

Thus,

$$\sum_{i=1}^n f\left(\frac{2i}{n}\right) = \sum_{i=1}^{n/2} 2\left(\frac{2i}{n}\right) + \sum_{i=n/2+1}^n 2.$$

Compute each sum:

$$\sum_{i=1}^{n/2} \frac{4i}{n} = \frac{4}{n} \cdot \frac{(n/2)(n/2+1)}{2} = \frac{n}{2} + 1,$$

$$\sum_{i=n/2+1}^n 2 = 2 \cdot \frac{n}{2} = n.$$

Therefore,

$$\sum_{i=1}^n f\left(\frac{2i}{n}\right) = \frac{n}{2} + 1 + n = \frac{3n}{2} + 1.$$

Multiply by Δx :

$$\frac{2}{n} \left(\frac{3n}{2} + 1 \right) = 3 + \frac{2}{n}.$$

Taking the limit,

$$\int_0^2 f(x) dx = \lim_{n \rightarrow \infty} \left(3 + \frac{2}{n} \right) = 3.$$