

Question 9

Find the exact area of the region enclosed by the curves

$$y = \ln(1 + x) \quad \text{and} \quad y = \frac{x}{1 + x},$$

between $x = 0$ and $x = 1$.

Question 9 - Solution

On the interval $0 \leq x \leq 1$, we compare the curves. At $x = 0$,

$$\ln(1) = 0, \quad \frac{0}{1} = 0.$$

At $x = 1$,

$$\ln 2 > \frac{1}{2},$$

so

$$\ln(1+x) \geq \frac{x}{1+x} \quad \text{on } [0, 1].$$

Thus the enclosed area is

$$A = \int_0^1 \left(\ln(1+x) - \frac{x}{1+x} \right) dx.$$

Split the integral:

$$A = \int_0^1 \ln(1+x) dx - \int_0^1 \frac{x}{1+x} dx.$$

Evaluate the first integral:

$$\int \ln(1+x) dx = (1+x) \ln(1+x) - x,$$

so

$$\int_0^1 \ln(1+x) dx = [(1+x) \ln(1+x) - x]_0^1 = 2 \ln 2 - 1.$$

For the second integral, rewrite:

$$\frac{x}{1+x} = 1 - \frac{1}{1+x}.$$

Then

$$\int_0^1 \frac{x}{1+x} dx = \int_0^1 \left(1 - \frac{1}{1+x} \right) dx = 1 - \ln 2.$$

Subtract:

$$A = (2 \ln 2 - 1) - (1 - \ln 2) = 3 \ln 2 - 2.$$

$$\boxed{3 \ln 2 - 2}$$

