

Question 8

Find the exact area of the region enclosed by the curves

$$y = e^{-x} \quad \text{and} \quad y = 1 + \left(\frac{1}{e} - 1\right)x.$$

Question 8 - Solution

First find the intersection points:

$$e^{-x} = 1 + \left(\frac{1}{e} - 1\right)x.$$

At $x = 0$,

$$e^0 = 1 = 1 + \left(\frac{1}{e} - 1\right) \cdot 0.$$

At $x = 1$,

$$e^{-1} = \frac{1}{e} = 1 + \left(\frac{1}{e} - 1\right) \cdot 1.$$

So the curves intersect at $x = 0$ and $x = 1$.

Since e^{-x} is convex, the chord (the line) lies above e^{-x} on $(0, 1)$, so the area is

$$A = \int_0^1 \left(1 + \left(\frac{1}{e} - 1\right)x - e^{-x}\right) dx.$$

Compute each piece:

$$\int_0^1 \left(1 + \left(\frac{1}{e} - 1\right)x\right) dx = \left[x + \left(\frac{1}{e} - 1\right)\frac{x^2}{2}\right]_0^1 = 1 + \frac{1}{2}\left(\frac{1}{e} - 1\right) = \frac{1}{2} + \frac{1}{2e},$$

$$\int_0^1 e^{-x} dx = [-e^{-x}]_0^1 = 1 - \frac{1}{e}.$$

Thus

$$A = \left(\frac{1}{2} + \frac{1}{2e}\right) - \left(1 - \frac{1}{e}\right) = -\frac{1}{2} + \frac{3}{2e} = \frac{1}{2}\left(\frac{3}{e} - 1\right).$$

$$\boxed{\frac{1}{2}\left(\frac{3}{e} - 1\right)}$$

