

Question 1

Find the exact area of the region enclosed by the curves

$$y = x\sqrt{4 - x^2} \quad \text{and} \quad y = x^2.$$

Question 1 - Solution

First find the points of intersection:

$$x\sqrt{4-x^2} = x^2.$$

Factor:

$$x(\sqrt{4-x^2} - x) = 0.$$

Thus,

$$x = 0 \quad \text{or} \quad \sqrt{4-x^2} = x.$$

Square the second equation:

$$4 - x^2 = x^2 \quad \Rightarrow \quad x^2 = 2 \quad \Rightarrow \quad x = \sqrt{2}.$$

On the interval $[0, \sqrt{2}]$, we have

$$x\sqrt{4-x^2} \geq x^2.$$

Therefore, the area is

$$A = \int_0^{\sqrt{2}} (x\sqrt{4-x^2} - x^2) dx.$$

Split the integral:

$$A = \int_0^{\sqrt{2}} x\sqrt{4-x^2} dx - \int_0^{\sqrt{2}} x^2 dx.$$

For the first integral, use substitution. Let

$$u = 4 - x^2, \quad du = -2x dx.$$

Then

$$\int x\sqrt{4-x^2} dx = -\frac{1}{2} \int u^{1/2} du = -\frac{1}{3} u^{3/2} = -\frac{1}{3} (4-x^2)^{3/2}.$$

Evaluate from 0 to $\sqrt{2}$:

$$\left[-\frac{1}{3} (4-x^2)^{3/2} \right]_0^{\sqrt{2}} = -\frac{1}{3} (2\sqrt{2}) + \frac{1}{3} (8) = \frac{8-2\sqrt{2}}{3}.$$

For the second integral:

$$\int_0^{\sqrt{2}} x^2 dx = \left[\frac{x^3}{3} \right]_0^{\sqrt{2}} = \frac{2\sqrt{2}}{3}.$$

Subtract:

$$A = \frac{8-2\sqrt{2}}{3} - \frac{2\sqrt{2}}{3} = \frac{8-4\sqrt{2}}{3}.$$

$$\boxed{\frac{8-4\sqrt{2}}{3}}$$