

Question 10

Evaluate the integral

$$\int \frac{e^x}{\sqrt{1 + e^{2x}}} dx.$$

Question 10 - Solution

Begin with an exponential substitution. Let

$$u = e^x.$$

Then

$$du = e^x dx.$$

Substitute into the integral:

$$\int \frac{e^x}{\sqrt{1+e^{2x}}} dx = \int \frac{1}{\sqrt{1+u^2}} du.$$

Inverse trigonometric step (expanded):

We recognize the integrand as the derivative of $\operatorname{arsinh} u$ or $\arctan u$ -type expressions. Use a trigonometric substitution to justify the result.

Let

$$u = \tan \theta.$$

Then

$$du = \sec^2 \theta d\theta, \quad \sqrt{1+u^2} = \sqrt{1+\tan^2 \theta} = \sec \theta.$$

Substitute:

$$\int \frac{1}{\sqrt{1+u^2}} du = \int \frac{\sec^2 \theta}{\sec \theta} d\theta = \int \sec \theta d\theta.$$

Recall the standard integral:

$$\int \sec \theta d\theta = \ln |\sec \theta + \tan \theta|.$$

Substitute back using

$$\tan \theta = u, \quad \sec \theta = \sqrt{1+u^2}.$$

Thus,

$$\int \frac{1}{\sqrt{1+u^2}} du = \ln(u + \sqrt{1+u^2}).$$

Substitute back $u = e^x$:

$$\boxed{\ln(e^x + \sqrt{1+e^{2x}}) + C}$$