

Question 7

Evaluate the integral

$$\int \sqrt{x^2 + x} \, dx.$$

Question 7 - Solution

Begin by completing the square:

$$x^2 + x = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4}.$$

Let

$$u = x + \frac{1}{2}.$$

Then

$$du = dx, \quad \sqrt{x^2 + x} = \sqrt{u^2 - \frac{1}{4}}.$$

The integral becomes

$$\int \sqrt{u^2 - \frac{1}{4}} du.$$

Now use a standard substitution for expressions of the form $\sqrt{u^2 - a^2}$. Let

$$u = \frac{1}{2} \sec \theta.$$

Then

$$du = \frac{1}{2} \sec \theta \tan \theta d\theta, \quad \sqrt{u^2 - \frac{1}{4}} = \frac{1}{2} \tan \theta.$$

Substitute:

$$\int \sqrt{u^2 - \frac{1}{4}} du = \int \frac{1}{2} \tan \theta \cdot \frac{1}{2} \sec \theta \tan \theta d\theta = \frac{1}{4} \int \tan^2 \theta \sec \theta d\theta.$$

Use the identity $\tan^2 \theta = \sec^2 \theta - 1$:

$$\frac{1}{4} \int (\sec^3 \theta - \sec \theta) d\theta.$$

Integrate:

$$\int \sec^3 \theta d\theta = \frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|),$$

$$\int \sec \theta d\theta = \ln |\sec \theta + \tan \theta|.$$

Thus,

$$\frac{1}{4} \left[\frac{1}{2} \sec \theta \tan \theta - \frac{1}{2} \ln |\sec \theta + \tan \theta| \right].$$

Return to x using

$$\sec \theta = 2u = 2x + 1, \quad \tan \theta = 2\sqrt{x^2 + x}.$$

Final Answer:

$$\boxed{\frac{1}{2} \left(x + \frac{1}{2}\right) \sqrt{x^2 + x} - \frac{1}{8} \ln |2x + 1 + 2\sqrt{x^2 + x}| + C}$$