

Question 6

Evaluate the integral

$$\int \sqrt{x^2 + x} \, dx.$$

Question 6 - Solution

Because the integrand involves a square root of a quadratic expression, an *Euler substitution* is effective.

Complete the square:

$$x^2 + x = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4}.$$

Use the substitution

$$x + \frac{1}{2} = \frac{1}{2} \left(t - \frac{1}{t}\right), \quad t > 0.$$

Then

$$dx = \frac{1}{2} \left(1 + \frac{1}{t^2}\right) dt,$$

and

$$\sqrt{x^2 + x} = \sqrt{\frac{1}{4} \left(t - \frac{1}{t}\right)^2 - \frac{1}{4}} = \frac{1}{2} \left(t - \frac{1}{t}\right).$$

Substitute into the integral:

$$\int \sqrt{x^2 + x} dx = \int \frac{1}{2} \left(t - \frac{1}{t}\right) \cdot \frac{1}{2} \left(1 + \frac{1}{t^2}\right) dt.$$

Simplify:

$$\frac{1}{4} \int \left(t - \frac{1}{t^3}\right) dt.$$

Integrate:

$$\frac{1}{4} \left(\frac{t^2}{2} + \frac{1}{2t^2}\right) = \frac{1}{8} \left(t^2 + \frac{1}{t^2}\right).$$

Substitute back using

$$t - \frac{1}{t} = 2 \left(x + \frac{1}{2}\right).$$

Thus,

$$\boxed{\frac{1}{2} \left(x + \frac{1}{2}\right) \sqrt{x^2 + x} - \frac{1}{8} \ln \left|2x + 1 + 2\sqrt{x^2 + x}\right| + C}$$