

Question 4

Evaluate the integral

$$\int \frac{x}{\sqrt{x^2 + 2x + 5}} dx.$$

Question 4 - Solution

Complete the square:

$$x^2 + 2x + 5 = (x + 1)^2 + 4.$$

Let

$$u = x + 1, \quad du = dx, \quad x = u - 1.$$

Then the integral becomes

$$\int \frac{u-1}{\sqrt{u^2+4}} du = \int \frac{u}{\sqrt{u^2+4}} du - \int \frac{1}{\sqrt{u^2+4}} du.$$

First integral: Let $w = u^2 + 4$, so $dw = 2u du$:

$$\int \frac{u}{\sqrt{u^2+4}} du = \frac{1}{2} \int w^{-1/2} dw = \sqrt{u^2+4}.$$

Second integral (expanded):

$$\int \frac{1}{\sqrt{u^2+4}} du.$$

Use a trigonometric substitution. Let

$$u = 2 \tan \theta.$$

Then

$$du = 2 \sec^2 \theta d\theta, \quad \sqrt{u^2+4} = \sqrt{4 \tan^2 \theta + 4} = 2 \sec \theta.$$

Substitute:

$$\int \frac{1}{\sqrt{u^2+4}} du = \int \frac{2 \sec^2 \theta}{2 \sec \theta} d\theta = \int \sec \theta d\theta.$$

Recall:

$$\int \sec \theta d\theta = \ln |\sec \theta + \tan \theta|.$$

Return to u :

$$\tan \theta = \frac{u}{2}, \quad \sec \theta = \frac{\sqrt{u^2+4}}{2}.$$

Thus,

$$\int \frac{1}{\sqrt{u^2+4}} du = \ln(u + \sqrt{u^2+4}).$$

Final Answer:

$$\boxed{\sqrt{x^2+2x+5} - \ln(x+1+\sqrt{x^2+2x+5}) + C}$$