

Question 3

Evaluate the integral

$$\int \frac{x^2}{\sqrt{x^2 - 9}} dx, \quad x > 3.$$

Question 3 - Solution

Because the integrand involves $\sqrt{x^2 - a^2}$, a *hyperbolic substitution* is natural.

Let

$$x = 3 \cosh t.$$

Then

$$dx = 3 \sinh t \, dt, \quad \sqrt{x^2 - 9} = \sqrt{9 \cosh^2 t - 9} = 3 \sinh t.$$

Substitute into the integral:

$$\int \frac{x^2}{\sqrt{x^2 - 9}} dx = \int \frac{9 \cosh^2 t}{3 \sinh t} \cdot 3 \sinh t \, dt = \int 9 \cosh^2 t \, dt.$$

Use the identity

$$\cosh^2 t = \frac{1 + \cosh(2t)}{2}.$$

Then

$$\int 9 \cosh^2 t \, dt = \frac{9}{2} \int (1 + \cosh(2t)) \, dt = \frac{9}{2} \left(t + \frac{1}{2} \sinh(2t) \right).$$

Simplify:

$$\frac{9}{2} t + \frac{9}{4} \sinh(2t).$$

Return to x . Since

$$x = 3 \cosh t \quad \Rightarrow \quad t = \operatorname{arcosh}\left(\frac{x}{3}\right),$$

and

$$\sinh(2t) = 2 \sinh t \cosh t = 2 \left(\frac{\sqrt{x^2 - 9}}{3} \right) \left(\frac{x}{3} \right) = \frac{2x\sqrt{x^2 - 9}}{9}.$$

Substitute back:

$$\boxed{\frac{9}{2} \operatorname{arcosh}\left(\frac{x}{3}\right) + \frac{x}{2} \sqrt{x^2 - 9} + C}$$