

Question 2

Evaluate the integral

$$\int \frac{x^2}{\sqrt{4-x^2}} dx.$$

Question 2 - Solution

Because the integrand contains $\sqrt{4-x^2}$, a trigonometric substitution is natural.

Let

$$x = 2 \sin \theta.$$

Then

$$dx = 2 \cos \theta \, d\theta, \quad \sqrt{4-x^2} = \sqrt{4-4\sin^2 \theta} = 2 \cos \theta.$$

Substitute into the integral:

$$\int \frac{x^2}{\sqrt{4-x^2}} dx = \int \frac{4 \sin^2 \theta}{2 \cos \theta} \cdot 2 \cos \theta \, d\theta = \int 4 \sin^2 \theta \, d\theta.$$

Use the identity

$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}.$$

Then

$$\int 4 \sin^2 \theta \, d\theta = 4 \int \frac{1 - \cos(2\theta)}{2} d\theta = 2 \int (1 - \cos(2\theta)) \, d\theta.$$

Integrate:

$$2 \left(\theta - \frac{1}{2} \sin(2\theta) \right) = 2\theta - \sin(2\theta).$$

Return to x using

$$\theta = \arcsin\left(\frac{x}{2}\right), \quad \sin(2\theta) = 2 \sin \theta \cos \theta = \frac{x}{2} \sqrt{4-x^2}.$$

Thus,

$$2 \arcsin\left(\frac{x}{2}\right) - \frac{x}{2} \sqrt{4-x^2} + C$$