

Question 7

Evaluate

$$\int \frac{x}{\sqrt{1-x^2}} \ln(1 + \sqrt{1-x^2}) \, dx.$$

Question 7 - Solution

Let

$$u = \sqrt{1 - x^2}.$$

Then

$$du = \frac{-x}{\sqrt{1 - x^2}} dx \quad \Rightarrow \quad \frac{x}{\sqrt{1 - x^2}} dx = -du.$$

So the integral becomes

$$- \int \ln(1 + u) du.$$

Now integrate by parts. Take

$$w = \ln(1 + u), \quad dv = du,$$

so

$$dw = \frac{1}{1 + u} du, \quad v = u.$$

Then

$$- \int \ln(1 + u) du = - \left(u \ln(1 + u) - \int \frac{u}{1 + u} du \right).$$

Simplify

$$\frac{u}{1 + u} = 1 - \frac{1}{1 + u},$$

hence

$$\int \frac{u}{1 + u} du = \int \left(1 - \frac{1}{1 + u} \right) du = u - \ln(1 + u).$$

Therefore,

$$- (u \ln(1 + u) - u + \ln(1 + u)) = -u \ln(1 + u) + u - \ln(1 + u) + C.$$

Substitute back $u = \sqrt{1 - x^2}$:

$$\boxed{-\sqrt{1 - x^2} \ln(1 + \sqrt{1 - x^2}) + \sqrt{1 - x^2} - \ln(1 + \sqrt{1 - x^2}) + C}$$