

Question 9

Evaluate the integral

$$\int \frac{x^4}{(1+x^2)^{3/2}} dx.$$

Question 9 - Solution

Use the trigonometric substitution

$$x = \tan \theta, \quad dx = \sec^2 \theta \, d\theta.$$

Then

$$1 + x^2 = \sec^2 \theta, \quad (1 + x^2)^{3/2} = \sec^3 \theta.$$

Substitute into the integral:

$$\int \frac{\tan^4 \theta}{\sec^3 \theta} \sec^2 \theta \, d\theta = \int \tan^4 \theta \cos \theta \, d\theta.$$

Rewrite $\tan^4 \theta$:

$$\tan^4 \theta = (\sec^2 \theta - 1)^2 = \sec^4 \theta - 2\sec^2 \theta + 1.$$

Thus,

$$\int (\sec^3 \theta - 2\sec \theta + \cos \theta) \, d\theta.$$

Integrate term by term:

$$\int \sec^3 \theta \, d\theta = \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln(\sec \theta + \tan \theta),$$

$$\int \sec \theta \, d\theta = \ln(\sec \theta + \tan \theta), \quad \int \cos \theta \, d\theta = \sin \theta.$$

Combining,

$$\frac{1}{2} \sec \theta \tan \theta - \frac{3}{2} \ln(\sec \theta + \tan \theta) + \sin \theta + C.$$

Substitute back:

$$\sec \theta = \sqrt{1 + x^2}, \quad \tan \theta = x, \quad \sin \theta = \frac{x}{\sqrt{1 + x^2}}.$$

Therefore,

$$\frac{x}{2} \sqrt{1 + x^2} - \frac{3}{2} \ln(x + \sqrt{1 + x^2}) + \frac{x}{\sqrt{1 + x^2}} + C$$