

Question 8

Evaluate the integral

$$\int (x^2 + 1) \ln(x^2 + 1) dx.$$

Question 8 - Solution

Using substitution does not work!

$$u = x^2 + 1, \quad du = 2x \, dx.$$

Rewrite the integrand:

$$(x^2 + 1) \ln(x^2 + 1) \, dx = u \ln u \cdot \frac{du}{2x}.$$

Instead, apply integration by parts directly.

Let

$$u = \ln(x^2 + 1), \quad dv = (x^2 + 1) \, dx.$$

Then

$$du = \frac{2x}{x^2 + 1} \, dx, \quad v = \frac{x^3}{3} + x.$$

Apply integration by parts:

$$\int (x^2 + 1) \ln(x^2 + 1) \, dx = \left(\frac{x^3}{3} + x \right) \ln(x^2 + 1) - \int \left(\frac{x^3}{3} + x \right) \frac{2x}{x^2 + 1} \, dx.$$

Simplify the remaining integral:

$$\left(\frac{x^3}{3} + x \right) \frac{2x}{x^2 + 1} = \frac{2x^4}{3(x^2 + 1)} + \frac{2x^2}{x^2 + 1}.$$

Use algebraic division:

$$\frac{x^4}{x^2 + 1} = x^2 - 1 + \frac{1}{x^2 + 1}.$$

Thus,

$$\int (x^2 + 1) \ln(x^2 + 1) \, dx = \left(\frac{x^3}{3} + x \right) \ln(x^2 + 1) - \int \left(\frac{2}{3}(x^2 - 1) + \frac{2}{3(x^2 + 1)} + \frac{2x^2}{x^2 + 1} \right) \, dx.$$

Compute term by term:

$$\begin{aligned} \int (x^2 - 1) \, dx &= \frac{x^3}{3} - x, \\ \int \frac{1}{x^2 + 1} \, dx &= \arctan x, \quad \int \frac{x^2}{x^2 + 1} \, dx = x - \arctan x. \end{aligned}$$

Putting everything together,

$$\boxed{\int (x^2 + 1) \ln(x^2 + 1) \, dx = \left(\frac{x^3}{3} + x \right) \ln(x^2 + 1) - \frac{2x^3}{9} - \frac{4x}{3} + \frac{4}{3} \arctan x + C}$$