

Question 6

Evaluate the integral

$$\int \frac{x^3 \ln x}{1+x^2} dx.$$

Question 6 - Solution

Rewrite the integrand by splitting the rational expression:

$$\frac{x^3}{1+x^2} = x - \frac{x}{1+x^2}.$$

Thus,

$$\int \frac{x^3 \ln x}{1+x^2} dx = \int x \ln x dx - \int \frac{x \ln x}{1+x^2} dx.$$

First integral: Use integration by parts with

$$u = \ln x, \quad dv = x dx.$$

Then

$$du = \frac{1}{x} dx, \quad v = \frac{x^2}{2}.$$

So

$$\int x \ln x dx = \frac{x^2}{2} \ln x - \frac{x^2}{4}.$$

Second integral: Use integration by parts with

$$u = \ln x, \quad dv = \frac{x}{1+x^2} dx.$$

Then

$$du = \frac{1}{x} dx, \quad v = \frac{1}{2} \ln(1+x^2).$$

Hence,

$$\int \frac{x \ln x}{1+x^2} dx = \frac{1}{2} \ln x \ln(1+x^2) - \frac{1}{2} \int \frac{\ln(1+x^2)}{x} dx.$$

The remaining integral is non-elementary and is expressed using the dilogarithm function:

$$\int \frac{\ln(1+x^2)}{x} dx = -\frac{1}{2} \text{Li}_2(-x^2).$$

Therefore,

$$\int \frac{x \ln x}{1+x^2} dx = \frac{1}{2} \ln x \ln(1+x^2) + \frac{1}{4} \text{Li}_2(-x^2).$$

Final Answer:

$$\boxed{\frac{x^2}{2} \ln x - \frac{x^2}{4} - \frac{1}{2} \ln x \ln(1+x^2) - \frac{1}{4} \text{Li}_2(-x^2) + C}$$