

Question 5

Evaluate

$$\int \frac{x^2}{\sqrt{x^2 + 1}} dx.$$

Question 5 - Solution**Step 1: Rewrite the integrand**

Write $x^2 = (x^2 + 1) - 1$:

$$\frac{x^2}{\sqrt{x^2 + 1}} = \frac{(x^2 + 1) - 1}{\sqrt{x^2 + 1}} = \sqrt{x^2 + 1} - \frac{1}{\sqrt{x^2 + 1}}.$$

Thus,

$$\int \frac{x^2}{\sqrt{x^2 + 1}} dx = \int \sqrt{x^2 + 1} dx - \int \frac{1}{\sqrt{x^2 + 1}} dx.$$

Step 2: Trigonometric substitution

Let

$$x = \tan \theta, \quad dx = \sec^2 \theta d\theta.$$

Then

$$\sqrt{x^2 + 1} = \sqrt{\tan^2 \theta + 1} = \sec \theta.$$

Step 3: Evaluate $\int \sqrt{x^2 + 1} dx$

Substitute:

$$\int \sqrt{x^2 + 1} dx = \int \sec \theta (\sec^2 \theta d\theta) = \int \sec^3 \theta d\theta.$$

To evaluate $\int \sec^3 \theta d\theta$, write

$$\sec^3 \theta = \sec \theta \sec^2 \theta.$$

Let

$$u = \sec \theta, \quad dv = \sec^2 \theta d\theta.$$

Then

$$du = \sec \theta \tan \theta d\theta, \quad v = \tan \theta.$$

By integration by parts,

$$\int \sec^3 \theta d\theta = \sec \theta \tan \theta - \int \tan \theta (\sec \theta \tan \theta) d\theta.$$

Simplify the integrand:

$$\tan^2 \theta = \sec^2 \theta - 1,$$

so

$$\int \sec^3 \theta d\theta = \sec \theta \tan \theta - \int \sec \theta (\sec^2 \theta - 1) d\theta.$$

Distribute:

$$= \sec \theta \tan \theta - \int \sec^3 \theta d\theta + \int \sec \theta d\theta.$$

Move the repeated integral to the left:

$$2 \int \sec^3 \theta d\theta = \sec \theta \tan \theta + \int \sec \theta d\theta.$$

Now compute $\int \sec \theta d\theta$:

$$\int \sec \theta d\theta = \int \frac{\sec \theta (\sec \theta + \tan \theta)}{\sec \theta + \tan \theta} d\theta.$$

Let

$$u = \sec \theta + \tan \theta, \quad du = (\sec \theta \tan \theta + \sec^2 \theta) d\theta = \sec \theta (\sec \theta + \tan \theta) d\theta.$$

Hence,

$$\int \sec \theta \, d\theta = \ln |\sec \theta + \tan \theta| + C.$$

Substitute back:

$$2 \int \sec^3 \theta \, d\theta = \sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|.$$

Therefore,

$$\int \sec^3 \theta \, d\theta = \frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) + C.$$

Returning to x :

$$\sec \theta = \sqrt{x^2 + 1}, \quad \tan \theta = x.$$

Thus,

$$\int \sqrt{x^2 + 1} \, dx = \frac{1}{2} \left(x\sqrt{x^2 + 1} + \ln(x + \sqrt{x^2 + 1}) \right) + C.$$

Step 4: Evaluate $\int \frac{1}{\sqrt{x^2 + 1}} \, dx$

Substitute again:

$$\int \frac{1}{\sqrt{x^2 + 1}} \, dx = \int \frac{1}{\sec \theta} (\sec^2 \theta \, d\theta) = \int \sec \theta \, d\theta.$$

From above,

$$\int \sec \theta \, d\theta = \ln |\sec \theta + \tan \theta| + C = \ln(x + \sqrt{x^2 + 1}) + C.$$

Step 5: Combine results

$$\int \frac{x^2}{\sqrt{x^2 + 1}} \, dx = \frac{1}{2} \left(x\sqrt{x^2 + 1} + \ln(x + \sqrt{x^2 + 1}) \right) - \ln(x + \sqrt{x^2 + 1}) + C.$$

Combine logarithms:

$$= \frac{x}{2} \sqrt{x^2 + 1} - \frac{1}{2} \ln(x + \sqrt{x^2 + 1}) + C.$$

Final Answer:

$$\boxed{\int \frac{x^2}{\sqrt{x^2 + 1}} \, dx = \frac{x}{2} \sqrt{x^2 + 1} - \frac{1}{2} \ln(x + \sqrt{x^2 + 1}) + C}$$

