

**Question 8**

Find the general antiderivative of

$$\int \left( \ln(x + \sqrt{1 + x^2}) + \frac{x^2}{\sqrt{1 + x^2}} \right) dx.$$

## Question 8 - Solution

Split the integral:

$$\int \ln(x + \sqrt{1+x^2}) \, dx + \int \frac{x^2}{\sqrt{1+x^2}} \, dx.$$

**Part A:**  $\int \ln(x + \sqrt{1+x^2}) \, dx$

Use integration by parts with

$$u = \ln(x + \sqrt{1+x^2}), \quad dv = dx, \quad v = x.$$

Let

$$g(x) = x + \sqrt{1+x^2}.$$

Then

$$du = \frac{g'(x)}{g(x)} \, dx, \quad g'(x) = 1 + \frac{x}{\sqrt{1+x^2}}.$$

Thus

$$du = \frac{1 + \frac{x}{\sqrt{1+x^2}}}{x + \sqrt{1+x^2}} \, dx = \frac{1}{\sqrt{1+x^2}} \, dx.$$

Therefore,

$$\int \ln(x + \sqrt{1+x^2}) \, dx = x \ln(x + \sqrt{1+x^2}) - \int \frac{x}{\sqrt{1+x^2}} \, dx.$$

Let  $w = 1 + x^2$ ,  $dw = 2x \, dx$ :

$$\int \frac{x}{\sqrt{1+x^2}} \, dx = \frac{1}{2} \int w^{-1/2} \, dw = \sqrt{1+x^2}.$$

Hence,

$$\int \ln(x + \sqrt{1+x^2}) \, dx = x \ln(x + \sqrt{1+x^2}) - \sqrt{1+x^2}.$$

**Part B:**  $\int \frac{x^2}{\sqrt{1+x^2}} \, dx$  (use an easier trigonometric substitution)

Let

$$x = \tan \theta, \quad dx = \sec^2 \theta \, d\theta, \quad \sqrt{1+x^2} = \sec \theta.$$

Then

$$\int \frac{x^2}{\sqrt{1+x^2}} \, dx = \int \frac{\tan^2 \theta}{\sec \theta} \sec^2 \theta \, d\theta = \int \tan^2 \theta \sec \theta \, d\theta.$$

Use  $\tan^2 \theta = \sec^2 \theta - 1$ :

$$\int \tan^2 \theta \sec \theta \, d\theta = \int (\sec^3 \theta - \sec \theta) \, d\theta.$$

Evaluate each term directly:

$$\int \sec \theta \, d\theta = \ln |\sec \theta + \tan \theta|,$$

and for  $\int \sec^3 \theta \, d\theta$ , integrate by parts:

$$\int \sec^3 \theta \, d\theta = \frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|).$$

Thus,

$$\int (\sec^3 \theta - \sec \theta) \, d\theta = \frac{1}{2} \sec \theta \tan \theta - \frac{1}{2} \ln |\sec \theta + \tan \theta|.$$

Return to  $x$  using  $\tan \theta = x$ ,  $\sec \theta = \sqrt{1+x^2}$ :

$$\int \frac{x^2}{\sqrt{1+x^2}} dx = \frac{1}{2}x\sqrt{1+x^2} - \frac{1}{2}\ln\left(x + \sqrt{1+x^2}\right).$$

**Combine Parts A and B:**

$$x \ln\left(x + \sqrt{1+x^2}\right) - \sqrt{1+x^2} + \frac{1}{2}x\sqrt{1+x^2} - \frac{1}{2}\ln\left(x + \sqrt{1+x^2}\right) + C$$

