

**Question 5**

Find an antiderivative of

$$\int \left( \frac{1}{x} \arctan x + \frac{x^2}{1+x^2} \right) dx.$$

## Question 5 - Solution

Split the integral:

$$\int \left( \frac{1}{x} \arctan x + \frac{x^2}{1+x^2} \right) dx = \int \frac{\arctan x}{x} dx + \int \frac{x^2}{1+x^2} dx.$$

## 1) The rational part

Rewrite:

$$\frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2}.$$

So

$$\int \frac{x^2}{1+x^2} dx = \int \left( 1 - \frac{1}{1+x^2} \right) dx = x - \arctan x.$$

2) The non-elementary part  $\int \frac{\arctan x}{x} dx$ 

A common attempt is integration by parts, which produces

$$\int \frac{\arctan x}{x} dx = \arctan x \ln|x| - \int \frac{\ln|x|}{1+x^2} dx,$$

and then the substitution  $x = \tan \theta$  turns the remaining integral into

$$\int \frac{\ln|x|}{1+x^2} dx = \int \ln(\tan \theta) d\theta.$$

The key clarification is:

$\int \ln(\tan \theta) d\theta$  **is not an elementary antiderivative.** So any step claiming

$$\int \ln(\tan \theta) d\theta = \theta \ln(\tan \theta) - \tan \theta + C$$

is incorrect (differentiate it to check).

A correct closed form uses the dilogarithm function  $\text{Li}_2$ . Start from

$$\arctan x = \frac{1}{2i} (\ln(1+ix) - \ln(1-ix)).$$

Then

$$\int \frac{\arctan x}{x} dx = \frac{1}{2i} \int \frac{\ln(1+ix) - \ln(1-ix)}{x} dx.$$

Use the identity

$$\frac{d}{dx} \text{Li}_2(u) = -\ln(1-u) \frac{u'}{u},$$

so, for real  $x$ ,

$$\frac{d}{dx} \text{Li}_2(-ix) = -\frac{\ln(1+ix)}{x}, \quad \frac{d}{dx} \text{Li}_2(ix) = -\frac{\ln(1-ix)}{x}.$$

Therefore,

$$\int \frac{\ln(1+ix)}{x} dx = -\text{Li}_2(-ix) + C, \quad \int \frac{\ln(1-ix)}{x} dx = -\text{Li}_2(ix) + C,$$

and hence

$$\int \frac{\arctan x}{x} dx = \frac{1}{2i} (\text{Li}_2(ix) - \text{Li}_2(-ix)) + C.$$

**Final Answer**

Combine both parts:

$$\int \left( \frac{1}{x} \arctan x + \frac{x^2}{1+x^2} \right) dx = (x - \arctan x) + \frac{1}{2i} \left( \operatorname{Li}_2(ix) - \operatorname{Li}_2(-ix) \right) + C.$$

