

Question 2

Find the most general antiderivative of

$$\int \left(e^{2x} \sin x + \frac{1}{(1-x)^2} \right) dx.$$

Question 2 - Solution

We evaluate each term separately.

First term:

$$\int e^{2x} \sin x \, dx.$$

Use integration by parts. Let

$$u = \sin x, \quad dv = e^{2x} \, dx.$$

Then

$$du = \cos x \, dx, \quad v = \frac{1}{2}e^{2x}.$$

So,

$$\int e^{2x} \sin x \, dx = \frac{1}{2}e^{2x} \sin x - \frac{1}{2} \int e^{2x} \cos x \, dx.$$

Apply integration by parts again to the remaining integral. Let

$$u = \cos x, \quad dv = e^{2x} \, dx.$$

Then

$$du = -\sin x \, dx, \quad v = \frac{1}{2}e^{2x}.$$

Thus,

$$\int e^{2x} \cos x \, dx = \frac{1}{2}e^{2x} \cos x + \frac{1}{2} \int e^{2x} \sin x \, dx.$$

Substitute this back:

$$\int e^{2x} \sin x \, dx = \frac{1}{2}e^{2x} \sin x - \frac{1}{4}e^{2x} \cos x - \frac{1}{4} \int e^{2x} \sin x \, dx.$$

Add $\frac{1}{4} \int e^{2x} \sin x \, dx$ to both sides:

$$\frac{5}{4} \int e^{2x} \sin x \, dx = \frac{1}{2}e^{2x} \sin x - \frac{1}{4}e^{2x} \cos x.$$

Solve:

$$\int e^{2x} \sin x \, dx = \frac{e^{2x}}{5} (2 \sin x - \cos x).$$

Second term:

$$\int \frac{1}{(1-x)^2} \, dx.$$

Let

$$u = 1 - x, \quad du = -dx.$$

Then

$$\int \frac{1}{(1-x)^2} \, dx = - \int u^{-2} \, du = u^{-1} = \frac{1}{1-x}.$$

Final Answer:

$$\boxed{\frac{e^{2x}}{5} (2 \sin x - \cos x) + \frac{1}{1-x} + C}$$