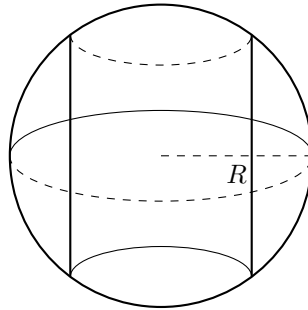


Question 3

A right circular cylinder is inscribed inside a sphere of radius R .



Find the dimensions (radius and height) of the cylinder that ****maximize its volume****, and express the maximum volume in terms of R .

Question 3 - Solution

Let the radius of the cylinder be r and height h . Since the cylinder fits in the sphere of radius R , we use the Pythagorean relation from the triangle formed by r , $h/2$, and R :

$$r^2 + \left(\frac{h}{2}\right)^2 = R^2 \Rightarrow h = 2\sqrt{R^2 - r^2}$$

Volume of the Cylinder:

$$V = \pi r^2 h = \pi r^2 \cdot 2\sqrt{R^2 - r^2} = 2\pi r^2 \sqrt{R^2 - r^2}$$

Maximize Volume:

Let:

$$V(r) = 2\pi r^2 \sqrt{R^2 - r^2}$$

Take the derivative (using product and chain rule):

$$\begin{aligned} V'(r) &= 2\pi \left[2r\sqrt{R^2 - r^2} + r^2 \cdot \frac{1}{2}(R^2 - r^2)^{-1/2}(-2r) \right] \\ &= 2\pi \left[2r\sqrt{R^2 - r^2} - \frac{r^3}{\sqrt{R^2 - r^2}} \right] \end{aligned}$$

Set $V'(r) = 0$:

$$2r\sqrt{R^2 - r^2} = \frac{r^3}{\sqrt{R^2 - r^2}} \Rightarrow 2(R^2 - r^2) = r^2 \Rightarrow 2R^2 - 2r^2 = r^2 \Rightarrow 3r^2 = 2R^2 \Rightarrow r^2 = \frac{2}{3}R^2$$

$$r = R\sqrt{\frac{2}{3}}, \quad h = 2\sqrt{R^2 - r^2} = 2\sqrt{R^2 - \frac{2}{3}R^2} = 2\sqrt{\frac{1}{3}R^2} = 2R\sqrt{\frac{1}{3}}$$

Maximum Volume:

$$V = \pi r^2 h = \pi \cdot \frac{2}{3}R^2 \cdot 2R\sqrt{\frac{1}{3}} = \frac{4\pi R^3}{3} \cdot \frac{1}{\sqrt{3}} = \frac{4\pi R^3}{3\sqrt{3}}$$

$$\boxed{\begin{aligned} r &= R\sqrt{\frac{2}{3}} \\ h &= 2R\sqrt{\frac{1}{3}} \\ V_{\max} &= \frac{4\pi R^3}{3\sqrt{3}} \end{aligned}}$$