

Question 8

Let

$$f(x) = \frac{x^2}{x^2 + 1}$$

- (a) Determine the intervals where the graph of f is concave up and concave down.
- (b) Find all inflection points, if any.

Question 8 - Solution

We are given

$$f(x) = \frac{x^2}{x^2 + 1}$$

First derivative

Using the quotient rule,

$$f'(x) = \frac{(2x)(x^2 + 1) - x^2(2x)}{(x^2 + 1)^2} = \frac{2x}{(x^2 + 1)^2}.$$

Second derivative

Differentiating $f'(x)$,

$$f''(x) = \frac{2(1 - 3x^2)}{(x^2 + 1)^3}.$$

(a) Concavity

Setting $f''(x) = 0$ gives

$$1 - 3x^2 = 0 \quad \Rightarrow \quad x = \pm \frac{1}{\sqrt{3}}.$$

For $|x| < \frac{1}{\sqrt{3}}$, the second derivative is positive, so the graph is concave up.

For $|x| > \frac{1}{\sqrt{3}}$, the second derivative is negative, so the graph is concave down.

Concave up on

$$\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

Concave down on

$$\left(-\infty, -\frac{1}{\sqrt{3}} \right) \cup \left(\frac{1}{\sqrt{3}}, \infty \right)$$

(b) Inflection points

Because the concavity changes at both values, inflection points occur at

$$x = \pm \frac{1}{\sqrt{3}}.$$

$$f\left(\pm \frac{1}{\sqrt{3}}\right) = \frac{1}{4}$$

Inflection points at

$$\left(-\frac{1}{\sqrt{3}}, \frac{1}{4} \right) \quad \text{and} \quad \left(\frac{1}{\sqrt{3}}, \frac{1}{4} \right)$$

Graph of $f(x)$ 