

Question 5

Let

$$f(x) = \frac{x}{x^2 + 1}$$

- (a) Determine the intervals where the graph of f is concave up and concave down.
- (b) Identify all inflection points.

Question 5 - Solution

We are given

$$f(x) = \frac{x}{x^2 + 1}$$

First derivative

Using the quotient rule,

$$f'(x) = \frac{(x^2 + 1) - 2x^2}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}.$$

Second derivative

Differentiating $f'(x)$,

$$f''(x) = \frac{-2x(3 - x^2)}{(x^2 + 1)^3}.$$

(a) Concavity

Setting $f''(x) = 0$ gives

$$x = -\sqrt{3}, \quad x = 0, \quad x = \sqrt{3}.$$

For $x < -\sqrt{3}$, the second derivative is negative, so the graph is concave down.

For $-\sqrt{3} < x < 0$, the second derivative is positive, so the graph is concave up.

For $0 < x < \sqrt{3}$, the second derivative is negative, so the graph is concave down.

For $x > \sqrt{3}$, the second derivative is positive, so the graph is concave up.

Concave up on

$$(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$$

Concave down on

$$(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$$

(b) Inflection points

Because the concavity changes at each critical value, inflection points occur at all three values.

$$f(0) = 0, \quad f(\pm\sqrt{3}) = \pm\frac{\sqrt{3}}{4}$$

Inflection points at

$$\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right), \quad (0, 0), \quad \left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$$

Graph of $f(x)$ 