

Question 9

Evaluate the limit:

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3}$$

Question 9 - Solution

As $x \rightarrow 0$, we have:

$$\tan x \rightarrow 0, \quad x \rightarrow 0 \quad \Rightarrow \quad \frac{\tan x - x}{x^3} \rightarrow \frac{0}{0} \quad (\text{indeterminate form})$$

Apply L'Hospital's Rule:

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} = \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{3x^2}$$

As $x \rightarrow 0$, $\sec^2 x \rightarrow 1$, so the numerator goes to 0, denominator to 0:

$$\Rightarrow \frac{0}{0} \quad \text{Apply L'Hospital's Rule again}$$

$$= \lim_{x \rightarrow 0} \frac{2 \sec^2 x \tan x}{6x} = \lim_{x \rightarrow 0} \frac{2 \cdot 1 \cdot 0}{6 \cdot 0} = \frac{0}{0} \quad (\text{Apply once more})$$

Take third derivative:

Numerator derivative:

$$\frac{d}{dx} [2 \sec^2 x \tan x] = 2 (2 \sec^2 x \tan^2 x + \sec^4 x)$$

Denominator derivative:

$$\frac{d}{dx} (6x) = 6$$

Now take the limit:

$$\lim_{x \rightarrow 0} \frac{2(2 \sec^2 x \tan^2 x + \sec^4 x)}{6} = \frac{2(2 \cdot 1 \cdot 0 + 1)}{6} = \frac{2}{6} = \boxed{\frac{1}{3}}$$