

Question 7

Evaluate the limit:

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3}$$

Question 7 - Solution

As $x \rightarrow 0$, we know:

$$\tan x \rightarrow 0, \quad x \rightarrow 0 \Rightarrow \tan x - x \rightarrow 0 \quad \text{and} \quad x^3 \rightarrow 0 \Rightarrow \text{Indeterminate form } \frac{0}{0}$$

Apply L'Hospital's Rule:

First derivative:

$$\lim_{x \rightarrow 0} \frac{\frac{d}{dx}[\tan x - x]}{\frac{d}{dx}[x^3]} = \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{3x^2}$$

As $x \rightarrow 0$, $\sec^2 x \rightarrow 1$, so numerator $\rightarrow 0$, denominator $\rightarrow 0$, still $\frac{0}{0}$

Second derivative:

$$\lim_{x \rightarrow 0} \frac{\frac{d}{dx}[\sec^2 x - 1]}{\frac{d}{dx}[3x^2]} = \lim_{x \rightarrow 0} \frac{2 \sec^2 x \cdot \sec x \tan x}{6x}$$

As $x \rightarrow 0$, $\tan x \rightarrow 0$, so numerator $\rightarrow 0$, denominator $\rightarrow 0$

Third derivative:

Differentiate numerator and denominator again. Let's simplify instead:

Use the series:

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \Rightarrow \tan x - x = \frac{x^3}{3} + \cdots$$

Then:

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} = \lim_{x \rightarrow 0} \frac{\frac{x^3}{3} + \cdots}{x^3} = \frac{1}{3}$$

Final Answer:

$$\boxed{\frac{1}{3}}$$