

Question 3

Evaluate the limit:

$$\lim_{x \rightarrow \infty} \left(x \ln \left(1 + \frac{1}{x} \right) \right)$$

Question 3 - Solution

We observe that as $x \rightarrow \infty$:

$$- \ln\left(1 + \frac{1}{x}\right) \rightarrow 0 \quad -x \rightarrow \infty$$

So the expression becomes:

$$\infty \cdot 0 \quad (\text{indeterminate form})$$

We rewrite it as a quotient:

$$x \ln\left(1 + \frac{1}{x}\right) = \frac{\ln\left(1 + \frac{1}{x}\right)}{1/x}$$

Now as $x \rightarrow \infty$, we get a $\frac{0}{0}$ form, so we can apply L'Hospital's Rule.

Let's compute the limit using L'Hospital:

$$\lim_{x \rightarrow \infty} \frac{\ln\left(1 + \frac{1}{x}\right)}{1/x} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} \ln\left(1 + \frac{1}{x}\right)}{\frac{d}{dx} \left(\frac{1}{x}\right)}$$

Differentiate numerator and denominator:

- Numerator derivative:

$$\frac{d}{dx} \ln\left(1 + \frac{1}{x}\right) = \frac{-1}{x^2\left(1 + \frac{1}{x}\right)} = \frac{-1}{x(x+1)}$$

- Denominator derivative:

$$\frac{d}{dx} \left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

Now plug in:

$$\begin{aligned} &= \lim_{x \rightarrow \infty} \frac{-1/(x(x+1))}{-1/x^2} = \lim_{x \rightarrow \infty} \frac{x^2}{x(x+1)} = \lim_{x \rightarrow \infty} \frac{x}{x+1} \\ &\quad \lim_{x \rightarrow \infty} \frac{x}{x+1} = 1 \end{aligned}$$

Final Answer:

$$\boxed{1}$$