

Question 9

Let $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, where $-1 < x < 1$.

- (a) Compute $f'(x)$.
- (b) Show that the result simplifies to a rational function.

Question 9 - Solution

We are given:

$$f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Let $u = \frac{2x}{1+x^2}$, so:

$$f'(x) = \frac{1}{\sqrt{1-u^2}} \cdot u'$$

Step 1: Differentiate $u = \frac{2x}{1+x^2}$:

Using the quotient rule:

$$u' = \frac{(1+x^2)(2) - 2x(2x)}{(1+x^2)^2} = \frac{2(1+x^2) - 4x^2}{(1+x^2)^2} = \frac{2-2x^2}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2}$$

Step 2: Simplify $\sqrt{1-u^2}$:

$$\begin{aligned} u = \frac{2x}{1+x^2} &\Rightarrow u^2 = \frac{4x^2}{(1+x^2)^2} \Rightarrow 1-u^2 = 1 - \frac{4x^2}{(1+x^2)^2} = \frac{(1+x^2)^2 - 4x^2}{(1+x^2)^2} \\ &= \frac{1+2x^2+x^4-4x^2}{(1+x^2)^2} = \frac{1-2x^2+x^4}{(1+x^2)^2} = \frac{(1-x^2)^2}{(1+x^2)^2} \Rightarrow \sqrt{1-u^2} = \frac{|1-x^2|}{1+x^2} \end{aligned}$$

Since $-1 < x < 1$, we have $1-x^2 > 0$, so:

$$\sqrt{1-u^2} = \frac{1-x^2}{1+x^2}$$

Step 3: Combine all parts:

$$f'(x) = \frac{1}{\sqrt{1-u^2}} \cdot u' = \frac{1}{\frac{1-x^2}{1+x^2}} \cdot \frac{2(1-x^2)}{(1+x^2)^2} = \frac{1+x^2}{1-x^2} \cdot \frac{2(1-x^2)}{(1+x^2)^2}$$

Cancel $1-x^2$ and one factor of $1+x^2$:

$$f'(x) = \frac{2}{1+x^2}$$

Final Answer:

$$f'(x) = \frac{2}{1+x^2}$$