

Question 7

Let

$$f(x) = \arcsin\left(\frac{2x}{\sqrt{1+x^2}+1}\right).$$

- (a) Determine the real domain.
- (b) Decide whether $f(x) = \arctan x$ holds on that domain. Justify your answer.
- (c) Compute $f'(x)$ wherever it exists.

Question 7 - Solution

Domain. Let $s = \sqrt{1+x^2} \geq 1$. The argument of arcsin must satisfy

$$\frac{4x^2}{(s+1)^2} = \frac{4(s-1)}{s+1} \leq 1.$$

This is equivalent to $s \leq 5/3$, hence

$$\boxed{D = [-4/3, 4/3]}.$$

Test the claimed identity. At $x = 4/3$, the arcsine is $\pi/2$, while $\arctan(4/3) < \pi/2$. Thus $f(x) = \arctan x$ is false.

Differentiate. For $|x| < 4/3$, put $u = 2x/(s+1)$. Then

$$u' = \frac{2}{s(s+1)}, \quad 1 - u^2 = \frac{5-3s}{s+1}.$$

Consequently

$$\boxed{f'(x) = \frac{2}{s\sqrt{s+1}\sqrt{5-3s}}}, \quad s = \sqrt{1+x^2}.$$

The derivative is defined on $(-4/3, 4/3)$ and grows without bound toward either endpoint.