

Question 5

Let

$$f(x) = \sec^{-1} \left(\frac{2x+3}{x-1} \right)$$

- (a) Determine the domain of $f(x)$.
- (b) Compute $f'(x)$.

Question 5 - Solution

(a) Domain:

The inverse secant function is defined for $|u| \geq 1$. Thus, we require

$$\left| \frac{2x+3}{x-1} \right| \geq 1$$

Squaring both sides:

$$\frac{(2x+3)^2}{(x-1)^2} \geq 1 \quad \Rightarrow \quad (2x+3)^2 \geq (x-1)^2$$

Expanding:

$$\begin{aligned} 4x^2 + 12x + 9 &\geq x^2 - 2x + 1 \\ 3x^2 + 14x + 8 &\geq 0 \end{aligned}$$

Factor:

$$(3x+2)(x+4) \geq 0$$

Critical points are $x = -4$ and $x = -\frac{2}{3}$. Since the quadratic opens upward, the inequality holds outside the interval between the roots:

$$x \leq -4 \quad \text{or} \quad x \geq -\frac{2}{3}$$

We must also exclude $x = 1$, where the denominator is zero.

Therefore, the domain of $f(x)$ is:

$$\boxed{(-\infty, -4] \cup [-\frac{2}{3}, 1) \cup (1, \infty)}$$

(b) Derivative:

Using the derivative formula:

$$\frac{d}{dx} (\sec^{-1}(u)) = \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$$

Let

$$u = \frac{2x+3}{x-1}$$

Then

$$\frac{du}{dx} = \frac{(x-1)(2) - (2x+3)(1)}{(x-1)^2} = \frac{-5}{(x-1)^2}$$

Substitute into the formula:

$$f'(x) = \frac{1}{\left| \frac{2x+3}{x-1} \right| \sqrt{\left(\frac{2x+3}{x-1} \right)^2 - 1}} \cdot \frac{-5}{(x-1)^2}$$

$$\boxed{f'(x) = \frac{-5}{(x-1)^2 \left| \frac{2x+3}{x-1} \right| \sqrt{\left(\frac{2x+3}{x-1} \right)^2 - 1}}}$$