

**Question 3**

Let

$$f(x) = \arcsin\left(\frac{2x}{1+x^2}\right), \quad x > 0.$$

- (a) Express  $f$  in terms of  $\arctan x$ , separately for  $0 < x \leq 1$  and  $x > 1$ .
- (b) Find  $f'(x)$  wherever it exists, and analyze  $x = 1$ .

**Question 3 - Solution**

Put  $t = \arctan x$ . For  $x > 0$ ,  $0 < t < \pi/2$  and

$$\frac{2x}{1+x^2} = \sin(2t).$$

The principal range of  $\arcsin$  is  $[-\pi/2, \pi/2]$ . Therefore

$$f(x) = \begin{cases} 2 \arctan x, & 0 < x \leq 1, \\ \pi - 2 \arctan x, & x > 1. \end{cases}$$

Differentiating on the two open intervals gives

$$f'(x) = \begin{cases} 2/(1+x^2), & 0 < x < 1, \\ -2/(1+x^2), & x > 1. \end{cases}$$

At  $x = 1$  the left and right derivatives are 1 and  $-1$ , respectively, so no derivative exists there.