

Question 2

Let

$$f(x) = \tan^{-1} \left(\frac{2x}{1-x^2} \right)$$

- (a) Show that $f(x) = 2 \tan^{-1}(x)$ for all $x \in (-1, 1)$
- (b) Use the identity in part (a) to find $f'(x)$.

Question 2 - Solution

(a) Use an identity for tangent of double angle:

We recall:

$$\tan(2\theta) = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

Let $\theta = \tan^{-1}(x) \Rightarrow \tan(\theta) = x$. Then:

$$\tan(2\theta) = \frac{2x}{1 - x^2} \Rightarrow 2\theta = \tan^{-1}\left(\frac{2x}{1 - x^2}\right)$$

So:

$$f(x) = \tan^{-1}\left(\frac{2x}{1 - x^2}\right) = 2 \tan^{-1}(x) \quad \text{for } x \in (-1, 1)$$

(b) Differentiate using identity:

From (a), $f(x) = 2 \tan^{-1}(x)$

Use the derivative:

$$\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1 + x^2} \Rightarrow f'(x) = 2 \cdot \frac{1}{1 + x^2} = \boxed{\frac{2}{1 + x^2}}$$