

Question 10

Let $a > 0$ be a real constant and define

$$f(x) = x^{ax}, \quad x > 0.$$

- (a) Use logarithmic differentiation to find $f'(x)$.
- (b) Simplify your answer as much as possible.
- (c) Find the second derivative $f''(x)$.
- (d) Determine all stationary points of $f(x)$.
- (e) Hence, determine the intervals on which $f(x)$ is increasing and decreasing.

Question 10 - Solution

We are given

$$f(x) = x^{ax}, \quad x > 0, \quad a > 0.$$

(a) Logarithmic Differentiation

Let

$$y = x^{ax}.$$

Taking natural logarithms:

$$\ln y = ax \ln x.$$

(b) First Derivative

Differentiate both sides with respect to x :

$$\frac{1}{y} \frac{dy}{dx} = a \ln x + a.$$

Multiply through by y :

$$\frac{dy}{dx} = y(a \ln x + a).$$

Substitute back $y = x^{ax}$:

$$f'(x) = x^{ax} a(\ln x + 1).$$

$$\boxed{f'(x) = ax^{ax}(\ln x + 1)}$$

(c) Second Derivative

Differentiate $f'(x)$ using the product rule:

$$f'(x) = ax^{ax}(\ln x + 1).$$

Let

$$u = x^{ax}, \quad v = \ln x + 1.$$

Then:

$$u' = ax^{ax}(\ln x + 1), \quad v' = \frac{1}{x}.$$

Thus,

$$f''(x) = a[u'v + uv'] = a \left[x^{ax} a(\ln x + 1)^2 + x^{ax} \frac{1}{x} \right].$$

Factor out x^{ax} :

$$f''(x) = ax^{ax} \left[a(\ln x + 1)^2 + \frac{1}{x} \right].$$

(d) Stationary Points

Stationary points occur when

$$f'(x) = 0.$$

Since $a > 0$ and $x^{ax} > 0$ for $x > 0$, we require:

$$\ln x + 1 = 0.$$

Solving:

$$\ln x = -1 \quad \Rightarrow \quad x = e^{-1}.$$

Thus, the only stationary point occurs at:

$$\boxed{x = \frac{1}{e}}.$$

(e) Increasing and Decreasing Behaviour

From

$$f'(x) = ax^{ax}(\ln x + 1),$$

the sign of $f'(x)$ depends on $\ln x + 1$.

- If $0 < x < \frac{1}{e}$, then $\ln x + 1 < 0 \Rightarrow f'(x) < 0$.
- If $x > \frac{1}{e}$, then $\ln x + 1 > 0 \Rightarrow f'(x) > 0$.

Decreasing on $(0, e^{-1})$, Increasing on (e^{-1}, ∞) .
